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Renormalization-group estimates of transport coefficients in the advection of a passive scalar by incompressible turbulence

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The advection of a passive scalar by incompressible turbulence is considered using recursive renormalization-group procedures in the differential subgrid shell thickness limit. It is shown explicitly that the higher-order nonlinearities induced by the recursive renormalization-group procedure preserve Galilean invariance. Differential equations, valid for the entire resolvable wave-number k range, are determined for the eddy viscosity and eddy diffusivity coefficients. It is shown that these higher-order nonlinearities do not contribute as $k \rightarrow 0$, but play an essential role as $k \rightarrow k_c$, the cutoff wave number separating the resolvable scales from the subgrid scales. The transport coefficients and the associated eddy Prandtl number are in good agreement with the k-dependent transport coefficients derived from closure theories and experiments.

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I. INTRODUCTION

The turbulent transport of a passive scalar, while serving sound pedagogical purposes, is also of interest in the spreading of temperature, humidity, and pollution in the atmosphere as well as in other problems [1]. Here we shall apply recursive renormalization-group (RNG) procedures to the subgrid modeling of a passive scalar field $T(\mathbf{k},t)$ being advected by a turbulent Navier-Stokes velocity field $\mathbf{u}(\mathbf{k},t)$. Subgrid modeling is necessary for the high-Reynolds-number turbulent flows of interest because of the limitations of current and forseeable supercomputers [2]. Another advantage of considering the problem of passive scalar transport is that the spectral transport coefficients (eddy diffusivity and eddy viscosity) determined from our RNG theory can be compared to those arising from closure-based theories [3,4]. It should be noted that the transport coefficients in these closure theories are determined over the whole resolvable scales.

Recently, two distinct approaches of RNG to fluid turbulence have arisen: one based on the work of Forster, Nelson, and Stephen [5], called ϵ-RNG, and the other based on the work of Rose [6], called recursive RNG. Some aspects of these two approaches have been discussed [7]. In particular, we point out here that in ϵ-RNG, a small parameter ϵ is introduced through the forcing correlation function. Yakhot and Orszag [8] had to extrapolate from $\epsilon \ll 1$ to $\epsilon \rightarrow 4$ in order to reproduce the Kolmogorov energy spectrum. Furthermore, it is also necessary to take the distant interaction limit [9], $k \rightarrow 0$. Thus it is difficult to compare the transport coefficients generated by Kraichnan [3] and Chollet [4] with that determined from ϵ-RNG.

In this paper, we continue our application of recursive RNG [10,11] to turbulence. The basic differences between the two RNG procedures are that in recursive RNG:

(1) The ϵ expansion is *not* applied.

(2) The turbulent transport coefficients are determined for the whole resolvable wave-number scales.

(3) Higher-order nonlinearities are generated in the renormalized momentum equation and play a critical role in determining the transport coefficients.

In Sec. II we derive the renormalized evolution equations for the passive scalar $T(\mathbf{k},t)$ and the fluid velocity $\mathbf{u}(\mathbf{k},t)$ as well as the recursion relations from which the eddy diffusivity and eddy viscosity can be determined. The turbulent transport coefficients for the second moments [i.e., for the time evolution of $U_{\alpha\beta}(\mathbf{k},t)=\langle u_\alpha(\mathbf{k},t)u_\beta(-\mathbf{k},t)\rangle$ and the scalar variance $\Theta(\mathbf{k},t)=\langle T(\mathbf{k},t)T(-\mathbf{k},t)\rangle$] are determined in Sec. III. In Sec. IV, we show that the higher-order RNG-induced nonlinearities do *not* contribute to the $k \rightarrow 0$ limit of the RNG recursion relations, but play a significant role for $k \rightarrow k_c$, where k_c is the wave number that separates the resolvable scale from the subgrid scale. It has been found to be very difficult to find fixed points (i.e., the transport coefficients) for the RNG difference recursion relations if the subgrid shell thickness is chosen too small [10,11]. If recursion RNG procedures are to be employed successfully in more complicated flow problems, then it is necessary that these difference recursion relations be simplified. Here, we overcome these difficulties by proceeding to the differential limit of these recursions relations, paying careful attention to the $k \rightarrow 0$ limit. Because of the presence of higher-order nonlinearities in the renormalized equations, it is not apparent that the Galilean invariance of the RNG model is still preserved. These questions are addressed in the Appendix, where we prove that the RNG evolution equations are indeed Galilean invariant, a property deemed necessary in any subgrid model [12]. The spectral eddy viscosity, diffusivity, and

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Prandtl number are derived in Sec. V, while in Sec. VI we present our conclusions.

II. RENORMALIZED MOMENTUM EQUATION FOR VELOCITY AND PASSIVE SCALAR

We consider a passive scalar $T(\mathbf{k},t)$ being advected by incompressible turbulence

$$\left[\frac{\partial}{\partial t}+\mu_0 k^2\right]T(\mathbf{k},t)=-ik_\alpha\int d^3j\, u_\alpha(\mathbf{k}-\mathbf{j},t)T(\mathbf{j},t)\ , \quad (1)$$

with the turbulent velocity field $\mathbf{u}(\mathbf{k},t)$ being determined from the Navier-Stokes equation

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha(\mathbf{k},t)=M_{\alpha\beta\gamma}(k)\int d^3j\, u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)+f_\alpha(\mathbf{k},t)\ . \quad (2)$$

Summation over repeated subscripts is understood, and

$$M_{\alpha\beta\gamma}(k)=[k_\beta D_{\alpha\gamma}(k)+k_\gamma D_{\alpha\beta}(k)]/2i$$

and (3)

$$D_{\alpha\beta}(k)=\delta_{\alpha\beta}-k_\alpha k_\beta/k^2\ .$$

Here μ_0 is the molecular diffusivity, ν_0 the molecular viscosity, and f_α is a random forcing term. The forcing correlation is given by

$$\langle f_\alpha(k,t)f_\beta(k',t')\rangle=D_0 k^{-y}D_{\alpha\beta}(k)\delta(k+k')\delta(t-t')\ , \quad (4)$$

where D_0 denotes the intensity of the forcing [8,11] and y is an appropriately chosen exponent so as to recover the Kolmogorov energy scaling in the inertial range ($y=3$). The dimension of D_0 is $[D_0]=L^2T^{-3}$ [13]. Since we are interested in the passive scalar advection by a velocity field, no forcing function is introduced in Eq. (1).

A. Outline of the recursive RNG procedure

Since the details of the recursive RNG procedure for Navier-Stokes turbulence have been presented before [6,10,11], we only briefly outline the steps here.

(1) The subgrid wave-number region (k_c,k_d) is partitioned into N shells

$$k_c\equiv k_N<k_{N-1}<\cdots<k_1<k_0\equiv k_d\ , \quad (5)$$

where k_c is the wave number separating the resolvable from the subgrid scales and k_d is at the order of Kolmogorov dissipation wave number. $k_n=f^n k_0$, $n=0,\ldots,N$, where f is a factor, $0<f<1$, measuring the coarseness of the subgrid partitioning. The limit $f\to1$ corresponds to a differential partitioning of the subgrid region ($N\to\infty$).

(2) The subgrid modes for the first shell, $k_1<k\le k_0$, are eliminated from the resolvable scale equation by the solution of the subgrid scale equation.

(3) A subgrid scale average is performed over the resultant resolvable scale equation. This will result not only in the introduction of the subgrid scale energy (or equivalently, forcing) spectrum, but it also results in a new triple nonlinearity and nonlocal eddy damping function in the resolvable momentum equation ($k\le k_c$).

(4) The above steps are repeated for each successive subgrid shell until all the subgrid scales have been removed.

(5) Since the subgrid scales evolve on a faster time scale than the resolvable scales, a multitime scale analysis can be performed to simplify the eddy damping function. The resultant eddy viscosity is a fixed point of an integrodifference recursion relation.

(6) The recursion relation for the eddy viscosity and the renormalized Navier-Stokes equation are rescaled.

It should be emphasized that there are two singular limits [7]: $f\to1$ and $k\to0$. A careful analysis must be done regarding these two limits and the associated averaging operations. We will address this issue in the present paper.

B. Distance Interaction Approximation $k\to0$

Consider the removal of the first subgrid shell and introduce the usual notation

$$u_\alpha(\mathbf{k},t)=\begin{cases}u_\alpha^>(\mathbf{k},t) & \text{if } k_1<k<k_0\\ u_\alpha^<(\mathbf{k},t) & \text{if } k<k_1\end{cases} \quad (6)$$

and

$$T(\mathbf{k},t)=\begin{cases}T^>(\mathbf{k},t) & \text{if } k_1<k<k_0\\ T^<(\mathbf{k},t) & \text{if } k<k_1\ .\end{cases} \quad (7)$$

We find that for $k<k_1$ the resolvable scale passive scalar and Navier-Stokes equation can be written as

$$\left[\frac{\partial}{\partial t}+\mu_0 k^2\right]T^<(\mathbf{k},t)=-k_\alpha\int d^3j[u_\alpha^<(\mathbf{k}-\mathbf{j},t)T^<(\mathbf{j},t)+u_\alpha^>(\mathbf{k}-\mathbf{j},t)T^<(\mathbf{j},t)+u_\alpha^<(\mathbf{k}-\mathbf{j},t)T^>(\mathbf{j},t)+u_\alpha^>(\mathbf{k}-\mathbf{j},t)T^>(\mathbf{j},t)] \quad (8)$$

and

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]\mu_\alpha^<(\mathbf{k},t)=f_\alpha^<(\mathbf{k},t)+M_{\alpha\beta\gamma}(k)\int d^3j[u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)+2u_\beta^>(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)+u_\beta^>(\mathbf{j},t)u_\gamma^>(\mathbf{k}-\mathbf{j},t)]\ . \quad (9)$$

The factor 2 in the Navier-Stokes Eq. (9) arises from the symmetry in the $\mathbf{j}\leftrightarrow\mathbf{k}-\mathbf{j}$ interchange.

We assume isotropy for both the velocity field and passive scalar, so that the subgrid velocity - passive scalar correlations are zero [14]:

$$\langle u^{>}T^{>}\rangle=0\ , \tag{10}$$

where $\langle\ \rangle$ represents averaging over the subgrid scales. The details of the implementation of the recursive RNG procedure to the advection of a passive scalar is a straightforward generalization of that for Navier-Stokes turbulence (see, e.g., [15]) and so will not be presented here.

Consider the resolvable scale Navier-Stokes equation, Eq. (9). The first and third terms on the right-hand side (rhs) of (9) are symmetric in j and $|\mathbf{k}-\mathbf{j}|$ in terms of their respective wave-number constraints in wave numbers. As a result, the distant interaction limit $k\to 0$ has no effect on the existence of these terms which will give rise to the standard quadratic nonlinearity and eddy viscosity, respectively. However, the second term on the rhs of (9) has the following constraint: j is in the subgrid while $|\mathbf{k}-\mathbf{j}|$ is in the resolvable scales. Specifically, the consistency condition requires that, for small k, j satisfies

$$j>k_c \quad\text{and}\quad j<k_c+kz\ ,$$

where $\mathbf{k}\cdot\mathbf{j}=kjz$. Since $|z|\le 1$, the range of integration must be $O(k)$.

Thus the second term on the rhs of Eq. (9) cannot contribute in the limit $k\to 0$ since the integrand is bounded. A similar conclusion can be drawn for the second and third terms in the renormalized passive scalar equation. Now it is well known [6,10,11] that the higher-order nonlinearities are induced by those terms under discussion. Since these terms are absent in the $k\to 0$ limit, we conclude that the higher-order nonlinearities will not contribute to the renormalized momentum equations and recursion relation for the transport coefficients in the distant interaction limit $k\to 0$. However, they will contribute to the renormalized Navier-Stokes and passive scalar equations for $0<k\le k_c$.

These conclusions can be tested directly using numerical simulation databases. Indeed, energy transfer and eddy viscosity can be analyzed using results from numerical simulations by introducing an artificial cut at a wave number k_c that is smaller than the maximum resolved wave number k_m of the simulation. With this fictitious separation between the subgrid and resolvable scales, it is possible to evaluate the effect of the subgrid $k_c<k<k_m$ on the resolvable scales $k<k_c$. To facilitate comparison with the recursive RNG analysis we consider separately the contribution to the energy transfer and eddy viscosity from the second and third term on the rhs of Eq. (9). We form an energy equation from the momentum equation and introduce the following notation: $T^{<<}(k)$ is the spectrum of energy transfer to mode $\mathbf{k}$ resulting from interactions between modes with wave numbers less than k_c; $T^{><}(k)$ and $T^{>>}(k)$ represent similar contributions from interactions with *one* or *both* modes above the cutoff k_c, respectively. The equivalent contributions to eddy viscosity in the energy equations are $\nu^{><}(k)=-T^{><}(k)/2k^2E(k)$ and $\nu^{>>}(k)=-T^{>>}(k)/2k^2E(k)$. To determine the behavior of the energy transfer and eddy viscosity $\nu^{><}(k)$ and $\nu^{>>}(k)$ we measured them in flow fields obtained from numerical simulations on 128^3 meshes of forced turbulence. The forced flow dataset was generated by Chasnov [16] in an large-eddy simulation (LES) of the Kolmogorov inertial range, using a subgrid model derived from the stochastic equation that is consistent with Eddy-damped-quasinormal Markovian (EDQNM) approximation [17,18].

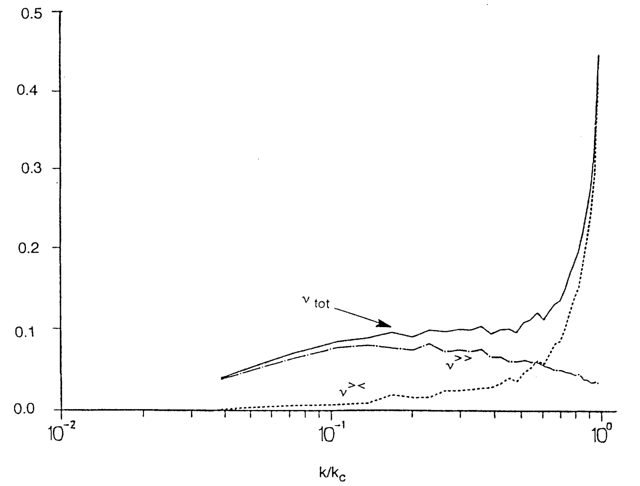

FIG. 1. Forced eddy viscosity profiles as determined directly from LES databases for the fluid velocity at one time instant. $\nu_{\rm tot}(k)=\nu^{>>}(k)+\nu^{><}(k)$, where $\nu^{>>}(k)$ arises from measured LES nonlocal subgrid energy transfer, and $\nu^{><}(k)$ arises from measured LES local subgrid energy transfer. It is important to note that $\nu^{><}(k)\to 0$ as $k\to 0$ and that $\nu^{><}(k)$ arises from the $u^{<}-u^{>}$ interaction. Note also the cusp behavior in $\nu^{><}(k)$ as $k\to k_c$.

In Fig. 1, we present a numerical measurement of $\nu^{><}(k)$. It demonstrates that the second term on the rhs of Eq. (9) does not contribute to the energy transfer process as $k\to 0$, consistent with our analysis.

C. Renormalized Navier-Stokes and passive scalar equations

After removing the nth subgrid shell, the renormalized passive scalar equation takes the form

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\mu_n(k)k^2\right]T^{<}(\mathbf{k},t)=&-ik_\alpha\int d^3j\,u_\alpha^{<}(\mathbf{k}-\mathbf{j},t)T^{<}(\mathbf{k},t)\\&-k_\alpha\sum_{h=1}^{n}\int d^3j\,d^3j'\frac{j_\beta}{\mu_{n-h}(j)j^2}u_\alpha^{<}(\mathbf{k}-\mathbf{j},t)u_\beta^{<}(\mathbf{j}-\mathbf{j}',t)T^{<}(\mathbf{j}',t)\\&-ik_\alpha\sum_{h=1}^{n}\int d^3j\,d^3j'\frac{M_{\alpha\beta\gamma}(\mathbf{k}-\mathbf{j})}{\nu_{n-h}(|\mathbf{k}-\mathbf{j}|)|\mathbf{k}-\mathbf{j}|^2}u_\beta^{<}(\mathbf{j}',t)u_\gamma^{<}(\mathbf{k}-\mathbf{j}-\mathbf{j}',t)T^{<}(\mathbf{j},t)\ .\end{aligned}\tag{11}$$

The restriction on the wave numbers are the following: $k_N < j < k_{N-1}$ in the second integral and $k_N < |\mathbf{k}-\mathbf{j}| < k_{N-1}$ in the third integral. The other wave-number constraints are as indicated by the superscript on the fields $\mathbf{u}$ and T.

The renormalized Navier-Stokes equation has the form [11]

$$\left[\frac{\partial}{\partial t}+\nu_n(k)k^2\right]u_\alpha^<(\mathbf{k},t)=f_\alpha^<(\mathbf{k},t)+M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)$$
$$+2M_{\alpha\beta\gamma}(\mathbf{k})\sum_{h=1}^{n}\int d^3j\, d^3j'\frac{1}{\nu_{n-h}(j)j^2}M_{\beta\beta'\gamma'}(j)u_{\beta'}^<(\mathbf{j}',t)u_{\gamma'}^<(\mathbf{j}-\mathbf{j}',t)u_\gamma^<(\mathbf{k}-\mathbf{j},t) , \quad (12)$$

where j is restricted to the subgrid shell in the second integral. Again, all other wave-number constraints are as indicated by the superscript.

D. Recursion relations for eddy viscosity and diffusivity

Although the second term on the rhs of Eq. (8) contributed a new triple nonlinear term, it does not make any contribution to the renormalized eddy diffusivity in the momentum equation in the process of removing the next subgrid shell. The reason is the following:

$$u_\alpha^>(\mathbf{k}-\mathbf{j},t)T^<(\mathbf{j},t)$$
$$\sim M_{\alpha\beta\gamma}(\mathbf{k}-\mathbf{j})\langle u_\beta^>(\mathbf{j}',t)u_\gamma^>(\mathbf{k}-\mathbf{j}-\mathbf{j}',t)\rangle T^<(\mathbf{j},t)\rightarrow 0 , \quad (13)$$

since the ensemble average will generate a delta function $\delta(\mathbf{k}-\mathbf{j})$ while $\mathbf{k}-\mathbf{j}$ is in the subgrid range. This is impossible, and so this second term cannot contribute to the eddy diffusivity.

After the removal of the $(n+1)$th subgrid shell, the sepctral eddy viscosity in the renormalized momentum equation is determined by the recursion relation [11]

$$\nu_{n+1}(k)=\nu_n(k)+\delta\nu_n(k) , \quad (14)$$

where

$$\delta\nu_n(k)=\frac{D_0}{k^2}\sum_{h=0}^{n}\int d^3j\frac{L(k,j,q)|\mathbf{k}-\mathbf{j}|^{-y}}{\nu_h(j)j^2\nu(|\mathbf{k}-\mathbf{j}|)|\mathbf{k}-\mathbf{j}|^2} \quad (15)$$

and

$$L(k,j,q)=-\frac{kj(1-z^2)[zq^2-kj]}{q^2} , \quad (16)$$

with $\mathbf{k}\cdot\mathbf{j}=kjz$ and $q=|\mathbf{k}-\mathbf{j}|$. This difference equation, after rescaling, has been solved by Zhou, Vahala, and Hossain [11] and fixed points were readily determined for finite $f\le 0.7$. However, it was very difficult to determine fixed points for finer subgrid partition factor $f>0.7$. In Sec. IV we shall pass to the differential subgrid limit $f\rightarrow 1$ and determine an ordinary differential equation (ODE) for the renormalized eddy viscosity over the entire resolvable scale which can be readily integrated.

In a similar fashion, the spectral eddy diffusivity in the renormalized passive scalar equation can be shown to be given by, after the removal of the $(n+1)$th subgrid shell,

$$\mu_{n+1}(k)=\mu_n(k)+\delta\mu_n(k) , \quad (17)$$

where

$$\delta\mu_n(k)=\frac{k_\alpha k_\beta}{k^2}\sum_{h=0}^{n}\int d^3j\frac{D_{\alpha\beta}(k-j)Q(|\mathbf{k}-\mathbf{j}|)}{\mu_{n-h}(j)j^2} , \quad (18)$$

with

$$\langle u_\beta^>(\mathbf{k},t)u_\gamma^>(\mathbf{k}',t)\rangle=CD_{\beta\gamma}(k)Q(k)(\mathbf{k}+\mathbf{k}') ,$$

with

$$Q(|\mathbf{k}|)=E(k)/4\pi k^2 .$$

C is a dimensional constant with the dimensions $[C]=1/T^2L^{(m-3)}$, where $m=\frac{5}{3}$ recovers the Kolmogorov energy spectrum. The renormalized eddy viscosity and diffusivity are defined as the fixed point of these recursion relations.

III. TURBULENT TRANSPORT COEFFICIENT IN THE SECOND MOMENTS

The concept of the spectral eddy viscosity and diffusivity are introduced in the second moments [3,4]. Thus the momentum equation spectral eddy viscosity and diffusivity are only a partial contribution to the total transport coefficients. Indeed, from our numerical measurement in Fig. 1, we expect that the triple nonlinear terms will contribute significantly to the energy transfer when k is near k_c.

The final renormalized passive scalar equation is

$$\left[\frac{\partial}{\partial t}+\mu(k)k^2\right]T(\mathbf{k},t)=-ik_\alpha\int d^3j\, u_\alpha(\mathbf{k}-\mathbf{j},t)T(\mathbf{k},t)$$
$$-\frac{k_\alpha}{\mu(k_c)k_c^{(m+1)/2}}\int d^3j\, d^3j'\frac{j_\beta}{j^{(3-m)/2}}u_\alpha(\mathbf{k}-\mathbf{j},t)u_\beta(\mathbf{j}-\mathbf{j}',t)T(\mathbf{j}',t)$$
$$-\frac{ik_\alpha}{\nu(k_c)k_c^{(m+1)/2}}\int d^3j\, d^3j'\frac{M_{\alpha\beta\gamma}(\mathbf{k}-\mathbf{j})}{|\mathbf{k}-\mathbf{j}|^{(3-m)/2}}u_\beta(\mathbf{j}',t)u_\gamma(\mathbf{k}-\mathbf{j}-\mathbf{j}',t)T(\mathbf{j},t) , \quad (19)$$

while the final renormalized Navier-Stokes equation is

$$\left[\frac{\partial}{\partial t}+\nu(k)k^2\right]u_\alpha(\mathbf{k},t)=f_\alpha(\mathbf{k},t)+M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)$$
$$+2\frac{1}{\nu(k_c)k_c^{(y+1)/3}}M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, d^3j'\frac{1}{j^{(5-y)/3}}M_{\beta\beta'\gamma'}(j)u_{\beta'}(\mathbf{j}',t)u_{\gamma'}(\mathbf{j}-\mathbf{j}',t)u_\gamma(\mathbf{k}-\mathbf{j},t)\ . \tag{20}$$

We consider the contribution of the triple nonlinear term in the renormalized eddy viscosity to the eddy viscosity first [19]. The second moment for the velocity field is defined as

$$U_{\alpha\beta}(\mathbf{k},t)\equiv\langle u_\alpha(\mathbf{k},t)u_\beta(-\mathbf{k},t)\rangle\ . \tag{21}$$

The time evolution of $U_{\alpha\beta}(\mathbf{k},t)$ is

$$\frac{\partial U_{\alpha\beta}(\mathbf{k},t)}{\partial t}=-2\nu(k)k^2U_{\alpha\beta}(\mathbf{k},t)+2\langle f_\alpha(\mathbf{k},t)u_\beta(-\mathbf{k},t)\rangle+T^D_{\alpha\beta}(\mathbf{k},t)+T^T_{\alpha\beta}(\mathbf{k},t)\ . \tag{22}$$

In this equation, $T^D_{\alpha\beta}(\mathbf{k},t)$ is the standard energy transfer from the quadratic nonlinearity. In contrast, $T^T_{\alpha\beta}(\mathbf{k},t)=-2\nu_T(k)k^2E(k)$ is the energy transfer arising from the RNG-induced triple nonlinearity. It is readily shown that [19]

$$\nu_T(k)=\frac{1}{2\nu(k_c)}\frac{1}{k^2}\times\int_{k_c}^{k+k_c}dj\,dz\frac{L(k,j,q)|\mathbf{k}-\mathbf{j}|^{-y-2}j^{y+1/3}}{\nu(\mathbf{k}-\mathbf{j})}\ . \tag{23}$$

In Fig. 2, we see that $\nu_T(k)$ is the major contributor to the cusplike behavior of the spectral eddy viscosity as $k\to 0$.

We now define $\Theta(k)$ as the scalar variance $\Theta(k)\equiv\langle T(-k,t)T(k,t)\rangle$. The dynamic equation for $\Theta(k)$ can be constructed from Eq. (19) by multiplying by $T(-k,t)$, followed by an average operation. Again, a quasinormal approximation is applied to reduce the fourth moment to the product of the second moments. Notice that the last term on the rhs of Eq. (19) will not contribute to the spectral equation since $\langle u_\beta(\mathbf{j}',t)u_\gamma(\mathbf{k}-\mathbf{j}-\mathbf{j}',t)\rangle\sim\delta(\mathbf{k}-\mathbf{j})$, a condition that cannot be satisfied since $\mathbf{k}-\mathbf{j}$ is in the subgrid scale.

The dynamical equation for the scalar variance is

$$\left[\frac{\partial}{\partial t}+\mu(k)k^2\right]\Theta(k,t)=\Sigma^D+\Sigma^T\ , \tag{24}$$

where Σ^D is the usual transfer function for the passive scalar. $\Sigma^T(k)$ is the additional contribution from the triple nonlinear term induced by the recursive RNG procedure

$$\Sigma^T(k)=-\frac{2k_\alpha}{\mu(k_c)}\int d^3j\,d^3j'\frac{j'_\beta}{j^{3-m/2}}\times\langle u_\alpha(\mathbf{k}-\mathbf{j})u_\beta(\mathbf{j}-\mathbf{j}')\rangle\times\langle T(\mathbf{j}')T(-\mathbf{k})\rangle=-2\mu_T(k)k^2\Theta(k,t)\ , \tag{25}$$

where

$$\mu_T(k)=\frac{1}{\mu(k_c)k_c^{m+1/2}}\frac{k_\alpha k_\beta}{k^2}\times\int d^3j\frac{D_{\alpha\beta}(k-j)Q(|\mathbf{k}-\mathbf{j}|)}{j^{3-m/2}} \tag{26}$$

and the incompressible condition has been used. It is seen in Fig. 3 that $\mu_T(k)$ is small when k is small. However, as $k\to k_c$, $\mu_T(k)$ increases rapidly.

The solution of $\nu_T(k)$ is very similar to that of $\mu_T(k)$ as $k\to k_c$. They are the major contributions to the strong cusp in the eddy viscosity found from the test field model [3] and EDQNM [4].

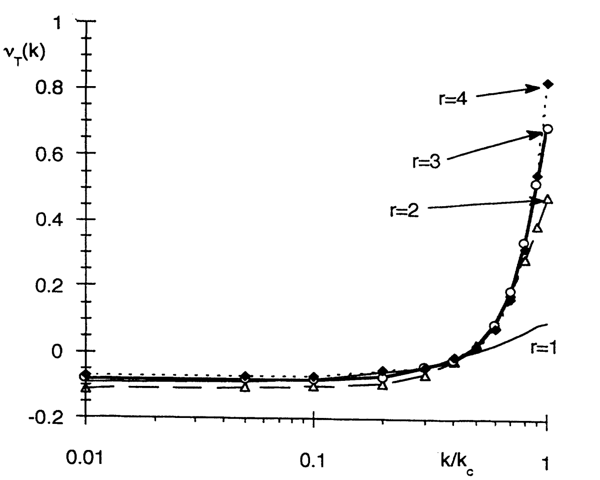

FIG. 2. The drain eddy viscosity $\nu_T(k)$ arising from the triple nonlinearities in the differential subgrid shell limit in recursive RNG. $r=k_c/K_p$ is a parameter in the production-type energy spectrum, so that $E(k)\to k^4$ as $k\to 0$. K_p is a parameter that controls the location of the peak in $E(k)$. As r increases, this peak in $E(k)$ moves to smaller k. Backscatter of energy from the subgrid scales to the larger spatial scales is seen for $k/k_c<0.4$, the region in which $\nu_T(k)<0$. For $r>1$, there is a sharp cusp as $k\to k_c$.

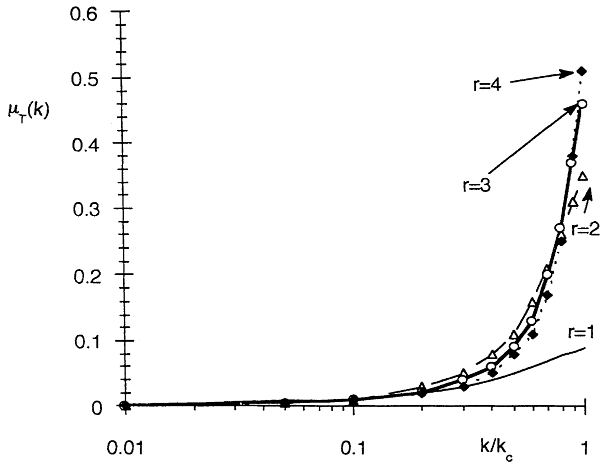

FIG. 3. The drain eddy diffusivity $\mu_T(k)$ arising from the triple nonlinearities in the differential subgrid shell limit of the scalar variance RNG evolution equation. The parameter r is as in Fig. 3. Notice that there is now no backscatter of scalar variance, since $\mu_T(k)$ is non-negative for all k. There is a strong cusp as $k \to k_c$.

Rose [6] discussed the role of the triple nonlinear terms in physical space. He pointed out that it represents the possibility of an exchange of scalar eddies between the resolvable and subgrid scales. This effects is an inherent property of measurements made on the passive scalar system with instruments which have a spatial resolution limited to an eddy width size greater than $1/k_c$.

IV. DIFFERENTIAL EQUATIONS FOR THE RENORMALIZED EDDY VISCOSITY AND DIFFUSIVITY

The differential limit $f \to 1$ is a singular one and this point has been discussed recently [7]. In particular, it is related to the assumption of local vs nonlocal interactions in k. In this section we will calculate the eddy viscosity and diffusivity under the differential equation limit for recursive RNG.

For recursive RNG we will find that the differential equations hold throughout the resolvable wave-number range $0 < k \leq k_c$. This should be contrasted with the ϵ-RNG eddy viscosity differential equation, which is valid only in the $k \to 0$ limit [8].

A. Rescaling of the recursion relation and momentum equations

From the self-similarity properties of the forcing and energy spectrum in the subgrid range, we expect that the viscosity ν_{n+1} to be simply related to ν_n for large n, while the diffusivity μ_{n+1} is simply related to μ_n. A rescaling can be performed on the recursion relation. In particular, consider

$$k \to k_{n+1}\tilde{k} \tag{27}$$

and define the dimensionless eddy viscosity $\tilde{\nu}_n(\tilde{k})$ and eddy diffusivity $\tilde{\mu}_n(\tilde{k})$

$$\tilde{\nu}_n(\tilde{k}) \equiv D_0^{-1/3} k_{n+1}^{(y+1)/3} \nu_n(k_{n+1}\tilde{k}) \quad \text{for } \tilde{k} \leq 1 , \tag{28}$$

$$\tilde{\mu}_n(\tilde{k}) \equiv C^{-1/2} k_{n+1}^{(m+1)/2} \mu_n(k_{n+1}\tilde{k}) \quad \text{for } \tilde{k} \leq 1 . \tag{29}$$

To recover the Kolmogorov energy spectrum, $y=3$ and $m=\frac{5}{3}$. Unless mentioned otherwise, for simplicity we will now drop the tilde notation. Hence $0 < k \leq 1$. The recursion relation for the eddy viscosity becomes

$$\nu_{n+1}(k) = f^{(y+1)/3}[\nu_n(fk) + \delta\nu_n(fk)] , \tag{30}$$

where

$$\delta\nu_n(fk) = f^{-(y+1)} \left[\frac{1}{k^2} \int d^3j \frac{L(k,j,q)|\mathbf{k}-\mathbf{j}|^{-y}}{\nu_n(fj)j^2 \nu_n(f|\mathbf{k}-\mathbf{j}|)|\mathbf{k}-\mathbf{j}|^2} + \sum_{h=1}^{n} \frac{1}{k^2} \int d^3j\, f^{-h(y+1)/3} \frac{L(k,j,q)|\mathbf{k}-\mathbf{j}|^{-y}}{\nu_{n-h}(f^{h+1}j)j^2 \nu_n(f|\mathbf{k}-\mathbf{j}|)|\mathbf{k}-\mathbf{j}|^2} \right] \tag{31}$$

and the summation term arises from the triple nonlinearity induced by the recursive RNG procedure. The recursion relation for the eddy diffusivity is

$$\mu_{n+1}(k) = f^{(m+1)/2}[\mu_n(fk) + \delta\mu_n(fk)] , \tag{32}$$

with

$$\delta\mu_n(fk) = \frac{f^{-(m+1)}}{4\pi} \left[\frac{k_\alpha k_\beta}{k^2} \int d^3j \frac{D_{\alpha\beta}(k-j)|\mathbf{k}-\mathbf{j}|^{-(m+2)}}{\mu_n(fj)j^2} + \sum_{h=1}^{n} \frac{k_\alpha k_\beta}{k^2} f^{-h(m+1)/2} \int d^3j \frac{D_{\alpha\beta}(k-j)|\mathbf{k}-\mathbf{j}|^{-(m+2)}}{\mu_{n-h}(f^{h+1}j)j^2} \right] , \tag{33}$$

where again the second term on the rhs of Eq. (33) arises from the induced triple nonlinearities. These equations are valid for any k in the resolvable scales: $0 < k \leq k_c$. In the limit $k \to 0$, the triple term contribution $\to 0$, as shown in Sec. II B.

B. Differential equation limit $f \to 1$

We now derive the differential equation from which the transport coefficients for finite k, $0<k\le 1$, are determined. The ODE in the distant interaction limit ($k=0$) will be derived in the next subsection. After the rescaling, we rewrite the recursion relation in the form

$$\nu_{n+1}(k)-f^{(y+1)/3}\nu_n(fk)=f^{(y+1)/3}\delta\nu_n(fk) \ . \tag{34}$$

For $f\to 1$, the number of interaction $n\to\infty$. Similarity consideration leads to

$$\nu_{n+1}(k)\to\nu(k), \quad n\to\infty \ . \tag{35}$$

Let $\Delta=1-f$. The lhs of Eq. (34) becomes

$$\nu(k)-[1-\Delta]^{(y+1)/3}\nu[k(1-\Delta)]\to\Delta\left[\frac{d\nu(k)}{dk}+\frac{y+1}{2}\nu(k)+O(\Delta)\right] . \tag{36}$$

As noted earlier [7], the partial average of Rose [6] must be employed in order to ensure the existence of the differential limit. The partial average is introduced since the distinction between the resolvable and subgrid scales become fuzzy in the limit of a differential subgrid partitioning $f\to 1$.

Following Rose [6], we first change the variable from j,z to j,q, with $dj\,dz=(q/kj)dj\,dq$, so that the rhs of Eq. (34) becomes

$$\begin{aligned}\delta\nu(k)=&\int dj\,dq\left[\frac{q}{kj}\right]\frac{L(k,j,q)}{\nu(j)k^2\nu(|\mathbf{k}-\mathbf{j}|)|\mathbf{k}-\mathbf{j}|^2|\mathbf{k}-\mathbf{j}|^y}\\ &+\int dj\,dq\left[\frac{q}{kj}\right]\frac{L(k,j,q)}{\nu(j)k^2\nu(|\mathbf{k}-\mathbf{j}|)|\mathbf{k}-\mathbf{j}|^2|\mathbf{k}-\mathbf{j}|^y}\left[\frac{j}{k_c}\right]^{(y+1)/3}\\ &\to\Delta\int_{1<q<1+k}dq\frac{L(k,1,q)}{k^3\nu^2(1)q^{y-1}}+\Delta\int_{1<j<1+k}dj\frac{L(k,j,1)}{k^3\nu^2(1)}j^{(y-2)/3} ,\end{aligned} \tag{37}$$

where

$$L(k,1,q)=k(1-z^2)(k-zq^2)/q^2 \tag{38}$$

and

$$L(k,j,1)=kj(1-z^2)(kj-z) \ . \tag{39}$$

As a result, the fixed-point renormalized eddy viscosity $\nu(k)$ is determined from the ODE at $O(\Delta)$,

$$k\frac{d\nu(k)}{dk}+\frac{y+1}{3}\nu(k)=\frac{1}{\nu^2(1)}[A_\nu(k)+B_\nu(k)] , \tag{40}$$

where

$$A_\nu(k)=\frac{1}{k^3}\int_1^{1+k}dq\frac{L(k,1,q)}{q^{y-1}} , \tag{41}$$

$$B_\nu(k)=\frac{1}{k^3}\int_1^{1+k}dj\,L(k,j,1)j^{(y-2)/3} \ . \tag{42}$$

Here z is evaluated at $j=1$ and $q=1$, respectively, in the $L(k,j,q)$ expression, Eq. (16).

The fixed-point ODE for the eddy diffusivity is given by

$$k\frac{d\mu(k)}{dk}+\frac{y+1}{3}\mu(k)=\frac{1}{\mu(1)}[A_\mu(k)+B_\mu(k)] , \tag{43}$$

where

$$A_\mu(k)=\frac{1}{2k}\int_{1\le q\le 1+k}dq\frac{\sin^2(k,q,1)}{q^{m+1}} , \tag{44}$$

$$B_\mu(k)=\frac{1}{2k}\int_{1\le j\le 1+k}dj\,\sin^2(k,1,j)j^{(m-1)/2} , \tag{45}$$

and $\sin^2(k,j,q)$ is the square of the sine of the angle defined by the k and q legs of the $(\mathbf{k},\mathbf{j},\mathbf{q})$ wave-vector triangle. Note that Eqs. (43)–(45) are identical with the eddy diffusivity ODE derived by Rose [6] for the advection of a passive scalar by a *prescribed frozen* velocity field.

C. Differential equations in the $k\to 0$ limit

In the $k\to 0$ limit, we have seen that the triple nonlinearities induced by RNG do not contribute to the eddy viscosity. As a result, the recursion relation will now contain only the usual quadratic contribution. We further simplify the analysis by taking the standard subgrid linear propagator [11] $G_h^{-1}(|\mathbf{k}-\mathbf{j}|)=[\partial/\partial t+\nu_h(|\mathbf{k}-\mathbf{j}|)]\sim G_h^{-1}(|\mathbf{j}|)$ as $k\to 0$.

The limits of the integration are given by

$$1<j-kz<1/f, \quad 1+kz<j<1/f+kz \ . \tag{46}$$

Thus the rhs of Eq. (34) becomes

$$\delta\nu_n(k)\equiv S-F-G , \tag{47}$$

where the integral limits for these terms are

$$\int_1^{1/f}dj\int_{-1}^{1}dz \quad \text{for } S , \tag{48}$$

$$\int_0^1 dz \int_1^{1+kz} dj \quad \text{for } F \ , \tag{49}$$

$$\int_{-1}^0 dz \int_{1/f+kz}^{1/f} dj \quad \text{for } G \ . \tag{50}$$

Terms F and G are the corrections to the symmetric term S. They are important for a finite bandwidth f. However, it is easy to show that $F+G=0$ for $f \to 1$ in the $k \to 0$ limit. Hence

$$\delta\nu_n^D(k) \to -\Delta \frac{1}{k\nu^2(1)} \left[\frac{1}{j^{y+1}} \right]_{j=1} \times \int_{-1}^1 dz[1-z^2] \left[z + \frac{k}{j}(yz^2-1) \right] = \Delta \frac{8}{15} \frac{1}{\nu^2(1)} \ , \tag{51}$$

while the lhs of Eq. (34) yields

$$k\frac{d\nu(k)}{dk} + \frac{y+1}{3}\nu(k) \to \frac{y+1}{3}\nu(k) \quad \text{as } k \to 0 \ ,$$

since $d\nu(k)/dk$ is bounded as $k \to 0$.

Thus, as $k \to 0$,

$$\nu(k \to 0) = \frac{3}{y+1} \frac{8}{15} \frac{1}{\nu^2(1)} \ . \tag{52}$$

A similar analysis can be performed on the fixed-point ODE for the eddy diffusivity, and this was not performed by Rose [6], who did not consider the $k \to 0$ limit carefully. Again, as $k \to 0$, the triple term will not contribute and we find that the corresponding $\delta\mu^D(k)$ term has the limiting form

$$\delta\mu^D(k) \to \Delta \frac{1}{2\mu(1)} \left[\frac{1}{j^{11/3}} \right]_{j=1} \int dz \left[1 + \frac{17}{3} \frac{kz}{j} \right] (1-z^2) = \Delta \frac{1}{2\mu(1)} \int_{-1}^1 dz(1-z^2) = \Delta \frac{2}{3\mu(1)} \tag{53}$$

as $k \to 0$. Hence, as $k \to 0$,

$$\mu(k \to 0) = \frac{2}{y+1} \frac{1}{\mu(1)} \ . \tag{54}$$

D. Momentum equation eddy viscosity and diffusivity

The ODE's, Eqs. (40) and (43), for the momentum equation eddy viscosity and diffusivity are readily solved and shown in Fig. 4. We observe that both the eddy viscosity and diffusivity have a similar plateau structure as $k \to 0$. Notice that the eddy diffusivity plateau is not obtained in the original numerical calculation of Rose [6] since Rose did not consider carefully the recursion relation as $k \to 0$. As $k \to k_c$, eddy viscosity displays a weak cusplike behavior while the eddy diffusivity decreases monotonically as $k \to k_c$. In this case, those curves are similar to that of Zhou, Vahala, and Hossain [11] and Rose [6].

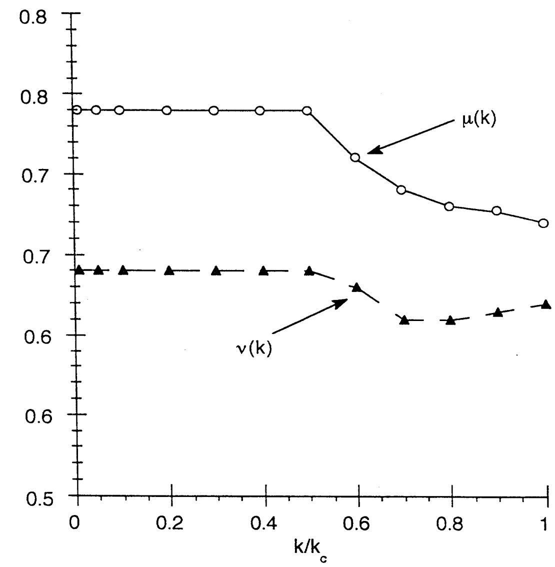

FIG. 4. A plot of the momentum eddy viscosity $\nu(k)$ and diffusivity $\mu(k)$ as a function of the resolvable scales, $0<k/k_c<1$. These profiles are determined from the ODE's for recursive RNG in the limit of differential subgrid shell thickness, $f \to 1$.

V. SPECTRAL EDDY VISCOSITY, DIFFUSIVITY, AND PRANDTL NUMBER

The spectral eddy viscosity is simply the sum of the contributions from the momentum equation and that of the effect of the RNG-induced triple nonlinear term in the energy equation. The result is presented in Fig. 5. It is apparent that our calculation is in qualitative agreement with that from the closure theory [3,4] and direct numerical measurements [16,19–21]. In particular, it

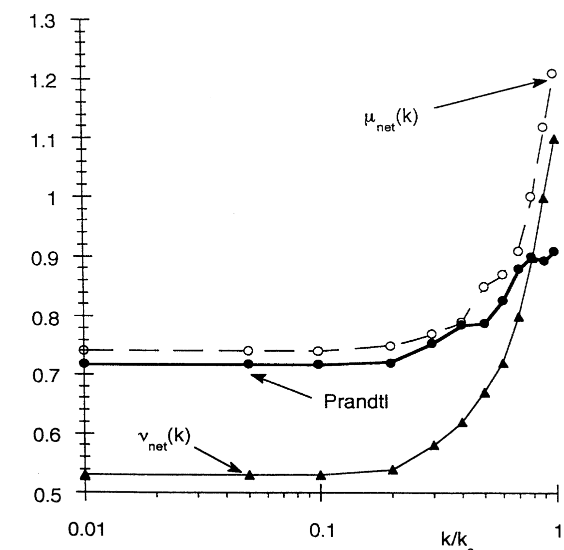

FIG. 5. The spectral eddy viscosity, diffusivity, and Prandtl number in the limit of differential subgrid shell thickness, $f \to 1$. The parameter $r=2$.

predicts the correct asymptotic behaviors of the eddy viscosity as $k \to 0$ and $k \to k_c$ [3].

Our spectral eddy diffusivity shows a plateau at $k \to 0$, in good agreement with the EDQNM calculation of Chollet [4]. However, the EDQNM calculation is not unique [4,14,22] and depends on the choice of the relaxation parameters $\tilde{\lambda}$, $\tilde{\lambda}'$, and $\tilde{\lambda}''$. Our diffusivity is in good agreement with EDQNM [4] with parameters chosen according to the direct interaction approximation (DIA) [23].

We now consider the eddy Prandtl number. Hinze [24] and Tennekes and Lumley [25] pointed out that the transfer of the passive scalar may be as effective as that of the velocity fields. Thus the turbulent Prandtl number is about 1. Fulachier and Dumas [26] carried out experiments in the turbulent boundary layer on a slightly heated plate in order to establish, mainly for the larger scales of motion, any analogy that may exist between the temperature and velocity fluctuations. Measurements were made in an open-return low-speed wind tunnel driven by a centrifugal blower. The turbulent boundary layer developed on the flat floor of the test section. The values of turbulent Prandtl numbers found experimentally [26] in the boundary layer are in the range of

$$0.6 \le \mathrm{Pr}^e \le 0.8 \ . \tag{55}$$

Herring *et al.* [22] compared—both analytically and numerically—two related spectral closures for the problem of decay of scalar fluctuations convected by isotropic turbulence. One was the test field model (TFM) [27,28] and the other was EDQNM. Lesieur and Chollet [29] and Herring *et al.* [22] found that the eddy Prandtl number

$$\mathrm{Pr}^e = \frac{\tilde{\lambda}' + \tilde{\lambda}''}{6\tilde{\lambda}} \ . \tag{56}$$

Equation (56) is the case in which the background energy spectrum is zero (for k less than some given wave number k_0). There is a corrective factor [22] if a more realistic assumption is made on the spectrum,

$$E(k) = \delta(k) + E'(k) \ , \qquad E'(k) = \begin{cases} k^n & \text{if } k \le k_0 \\ (k_0/k)^{-5/3} k_0^n & \text{if } k \ge k_0 \end{cases} \ . \tag{57}$$

The corrective factor Γ' ranges from $\frac{9}{5}$ (for $n = 1$) to $\frac{7}{5}$ (for $n = \infty$). Herring *et al.* [22] reported the work of Quarini, who studied the parameter range of $\tilde{\lambda}'/\tilde{\lambda}''$ which gives a Pr^e within the experimentally prescribed bounds, Eq. (55). Quarini's calculation indicates that the Lagrangian history direct interaction approximation (LHDIA) [30], $\tilde{\lambda}' = 0$, is the best choice ($\mathrm{Pr}^e = 0.6$). Herring *et al.* [22] also gives the correspondence between the EDQNM and TFM. Now in the TFM, there are three parameters g_v, g_θ, and $\tilde{g}_\theta$. They find that for correspondence between EDQNM and TFM,

$$2g_\theta^2 = \tilde{\lambda}'/\tilde{\lambda} \ , \tag{58a}$$

$$\tilde{g}_\theta^2 = \tilde{\lambda}''/\tilde{\lambda} \ . \tag{58b}$$

Substituting (58) into (56), we found that

$$\mathrm{Pr}^e = \frac{2g_\theta^2 + \tilde{g}_\theta^2}{6} \ .$$

When the TFM parameters are chosen according to the DIA ($g_\theta^2 = 0.5$, $\tilde{g}_\theta^2 = 1$), the eddy Prandtl number $\mathrm{Pr}^e = 0.33$. On the other hand, when the TFM parameters are chosen according to LHDIA ($g_\theta^2 = 0$, $\tilde{g}_\theta^2 = 3.61$), the eddy Prandtl number $\mathrm{Pr}^e = 0.6$.

The EDQNM *spectral* Prandtl number was investigated by Chollet [4]. He found that it depends on the choice of these EDQNM parameters [4,14]. In the LHDIA case, $\mathrm{Pr}^e(k)$ remains approximately equal to 0.6, even in the vicinity of wave-number cutoff k_c. This is very close to the finding of Herring *et al.* from the TFM and EDQNM. However, for the DIA case, $\mathrm{Pr}^e(k)$ has a plateau value of 0.33, with a cusplike behavior near k_c with $\mathrm{Pr}^e(k_c) = 0.6$.

Recently, the spectral Prandtl number was studied via the numerical simulations [20]. The large-eddy simulations, with subgrid model of spectral eddy viscosity and conductivity of Kraichnan [3] and Chollet [4], were performed at high Reynolds number. The highest resolution of Leisuer and Rogallo [20] is 128^3. This corresponds to a cutoff wave number $k_c = 64$. Using a fictitious cutoff wave number k_c' ($k_c' = 32$), Leisuer and Rogallo evaluated the kinetic energy and scalar transfers, resulting from triads k,p,q, where $k < k_c'$ and p or $q > k_c'$. These transfers were calculated directly from the simulated velocity and scalar fields, and when divided by $[E(k_c')/k_c']^{1/2}$, give, respectively, the spectral eddy viscosity and diffusivity. Lesieur and Rogallo [20] found their spectral Plandtl number $\mathrm{Pr}^e(k)$ only rose from 0.2 at small k to 0.8 near the cutoff. Lesieur [14] recently found that the turbulent Plandtl number may be much closer to 1 than that of Lesieur and Rogallo [20].

The spectral RNG Prandtl number can be easily determined from our calculated eddy viscosity and diffusivity (Fig. 5). It is a function k and has values ranging from 0.72 to 0.92. Note that our turbulent Prandtl number in the $k \to 0$ limit is very close to that reported by Yakhot and Orszag [8] (0.7179).

VI. SUMMARY AND DISCUSSION

In this paper we have applied recursive RNG to the problem of the advection of a passive scalar by incompressible turbulence. We have clarified the role of the higher-order RNG-induced nonlinearities and show the following: (a) The renormalized evolution equations are still Galilean invariant (i.e., these higher-order nonlinearities do not destroy the Galilean invariance of the original equations). This is an important property that needs to be preserved in subgrid modeling, especially as one proceeds to more complicated flows and boundaries. (b) These higher-order nonlinearities do not contribute to the transport coefficients as $k \to 0$.

Now the typical by-product of the recursive RNG methods is a complicated integrodifference recursion relation to be solved for the eddy transport coefficients [6,10,11]. This recursion relation is a function of the subgrid shell thickness parameter f. Here, we have shown how to pass to the differential subgrid shell thickness limit $f \rightarrow 1$. In this limit, we recover an ordinary differential equation for the eddy coefficients—an equation that is readily solved.

The ODE that is derived in recursive RNG is fundamentally different from that derived by ϵ-RNG techniques. In ϵ-RNG, one is forced into taking the $k \rightarrow 0$ limit [8,9], and the independent variable of the resulting ODE is actually the cutoff wave number k_c. In recursive RNG, the independent variable is the resolvable scale wave number k, $0<k\leq k_c$, with a renormalization transformation that permits k_c to be fixed. There is no renormalization transformation made in the Yakhot-Orszag ϵ-RNG formulation. In the limit $k \rightarrow 0$, the eddy transport coefficients from both theories are in very close agreement. This is to be expected since the higher-order recursive RNG induced nonlinearities$\rightarrow 0$ as $k \rightarrow 0$. The slight difference in the eddy coefficients (in the $k \rightarrow 0$) between the two theories can be attributed to the ϵ expansion procedure and to the treatment of k_c, i.e., whether one performs RNG rescaling transformations (recursive RNG) or not. The important effect of the triple nonlinearities introduced by the recursive RNG procedure are clearly seen in the second moment equations, especially for resolvable wave number $k \rightarrow k_c$.

The spectral eddy viscosity, diffusivity, and Prandtl number are determined and we find good agreement with both closure theory [3,4,14,22] and direct numerical simulations [16,19–21].

APPENDIX: GALILEAN INVARIANCE OF THE RENORMALIZED NAVIER-STOKES AND PASSIVE SCALAR EQUATIONS

In this Appendix, we turn our attention to the question of the Galilean invariance of the renormalized Navier-Stokes and passive scalar equations (19) and (20). The importance of Galilean invariance in turbulence modeling has been emphasized by Speziale [12]. To be consistent with the basic physics, it is required that the description of the turbulence be the same in all inertial frames of reference. The appearance of the triple nonlinear term, which is a function of the resolvable scales velocity fields, makes the property of the Galilean invariance of our recursive RNG procedure not apparent. We now show that both the renormalized Navier-Stokes equation and the renormalized passive scalar equation are Galilean invariant.

1. Galilean invariance in Navier-Stokes equation: Review

The Galilean transformation is

$$\mathbf{x} \rightarrow \mathbf{x}^* - \mathbf{U}_0 t^*, \quad t \rightarrow t^* .$$

Thus one has

$$\mathbf{u} = \mathbf{u}^* - \mathbf{U}_0, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial x^*},$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t^*} + \mathbf{U}_{0\beta}\frac{\partial}{\partial x_\beta} .$$

While the Galilean transformation for the Navier-Stokes equation in physical space is trivial, the Galilean transformation in wave-number space is less obvious, due to the lack of differential operations. For convenience, we first review how Galilean invariance is preserved for the Navier-Stokes equation in the wave-number space.

Under the Galilean transformation, the lhs of the Navier-Stokes equation [cf. Eq. (2)] becomes

$$\frac{\partial u^*_\alpha(\mathbf{k}^*,t)}{\partial t^*} + U_{0\beta} i k^*_\beta u^*_\alpha(\mathbf{k}^*,t) + \nu_0 k^{*2}[U_{0\alpha}\delta(\mathbf{k}^*) + u^*_\alpha(\mathbf{k}^*,t)] = \frac{\partial u^*_\alpha(\mathbf{k}^*,t)}{\partial t^*} + U_{0\beta} i k^*_\beta u^*_\alpha(\mathbf{k}^*,t) + \nu_0 k^{*2} u^*_\alpha(\mathbf{k}^*,t) , \tag{A1}$$

where in the last step, we have used the δ function property $k^{*2}\delta(\mathbf{k}^*)=0$.

Under the Galilean transformation, the rhs of the Navier-Stokes equation [cf. Eq. (2)] becomes

$$M_{\alpha\beta\gamma}(k^*)\int d^3j[u^*_\beta(\mathbf{j}^*,t) - U_{0\beta}\delta(\mathbf{j}^*)][u^*_\gamma(\mathbf{k}^*-\mathbf{j}^*,t) - U_{0\gamma}\delta(\mathbf{k}^*-\mathbf{j}^*)]$$
$$= M_{\alpha\beta\gamma}(k^*)\int d^3j^* u^*_\beta(\mathbf{j}^*,t)u^*_\gamma(\mathbf{k}^*-\mathbf{j}^*,t) + iU_{0\beta}k^*_\beta u^*_\alpha(\mathbf{k}^*,t) , \tag{A2}$$

where we have used the property of the δ function, the incompressible condition, and

$$M_{\alpha\beta\gamma}(k^*)U_{0\beta}u^*_\gamma(\mathbf{k}^*,t) = U_{0\beta}k^*_\beta u^*_\alpha(\mathbf{k}^*,t)/2i .$$

Thus, as expected, the Navier-Stokes equation is invariant under a Galilean transformation due to the cancellation of the second term on the rhs of Eqs. (A1) and (A2).

2. The renormalized Navier-Stokes equation under a Galilean transformation

To show that the renormalized Navier-Stokes equation is invariant under a Galilean transformation, we need only consider the recursive RNG-induced triple nonlinear term, denoted by $\mathcal{T}$:

$$\mathcal{T}\equiv 2M_{\alpha\beta\gamma}(k)\int d^3j\, d^3j' \left[\frac{j}{k_c}\right]^{4/3} \frac{M_{\beta\beta'\gamma'}(j)}{\nu(k_c)j^2} u_{\beta'}(\mathbf{j}-\mathbf{j}',t)u_{\gamma'}(\mathbf{j}',t)u_{\gamma}(\mathbf{k}-\mathbf{j},t) \ . \tag{A3}$$

It is important to note that j is in the subgrid.

Under a Galilean transformation, Eq. (A3) becomes

$$\mathcal{T}^*=2M_{\alpha\beta\gamma}(k^*)\int d^3j^* d^3j'^* \left[\frac{j^*}{k_c}\right]^{4/3} \frac{M_{\alpha\beta\gamma}(j^*)}{\nu(k_c^*)j^{*2}}[u^*_{\beta'}(\mathbf{j}^*-\mathbf{j}'^*,t)-U_{0\beta'}\delta(\mathbf{j}^*-\mathbf{j}^*)]$$
$$\times[u^*_{\gamma'}(\mathbf{j}'^*,t)-U_{0\gamma'}\delta(\mathbf{j})'^*][u^*_{\gamma}(\mathbf{k}^*-\mathbf{j}^*,t)-U_{0\gamma}\delta(\mathbf{k}^*-\mathbf{j}^*)] \ .$$

Since j^* is in the subgrid scale, while $\mathbf{j}'^*$ and k^* are in the supergrid, $\delta(\mathbf{k}^*-\mathbf{j}^*)$ and $\delta(\mathbf{j}^*-\mathbf{j}'^*)$ can never be simultaneously satisfied. As a result,

$$\mathcal{T}^*=2M_{\alpha\beta\gamma}(k^*)\int d^3j^* d^3j'^* \left[\frac{j^*}{k_c}\right]^{4/3} \frac{M_{\alpha\beta\gamma}(j^*)}{\nu(k_c^*)j^{*2}} u^*_{\beta'}(\mathbf{j}^*-\mathbf{j}'^*,t)[u^*_{\gamma'}(\mathbf{j}'^*,t)-U_{0\gamma'}\delta(\mathbf{j}'^*)]u^*_{\gamma}(\mathbf{k}^*-\mathbf{j}^*,t) \ . \tag{A4}$$

Now only one term in Eq. (A4) could violate the Galilean invariance of the renormalized Navier-Stokes equation. However,

$$\int d^3j'^*\delta(\mathbf{j}'^*)u^*_{\beta'}(\mathbf{j}^*-\mathbf{j}'^*,t)\rightarrow u^*_{\beta'}(\mathbf{j}^*,t) \ .$$

This is not permissable since $u^*_{\beta'}\equiv u^{<*}_{\beta'}$ and j^* is restricted to the subgrid. Thus $\mathcal{T}=\mathcal{T}^*$. Hence the triple term is Galilean invariant.

3. Galilean invariance in the renormalized passive scalar equation

The renormalized passive scalar equation has two triple nonlinear terms. The proof of the Galilean invariance of the renormalized passive scalar equation proceeds in a similar manner to that for the renormalized Navier-Stokes equation.

For the first triple nonlinear term, labeled $\mathcal{P}_1$ after a Galilean transformation we found

$$\mathcal{P}_1\sim\int d^3j^* d^3j'^*[u^{<*}_{\alpha}(\mathbf{k}^*-j^*,t)+U_{0\alpha}\delta(\mathbf{k}^*-\mathbf{j}^*)][u^{<*}_{\beta}(\mathbf{j}^*-\mathbf{j}'^*,t)+U_{0\beta}\delta(\mathbf{j}^*-\mathbf{j}'^*)]T^{<*}(\mathbf{j},t)$$
$$=\int d^3j^* d^3 u^{<*}_{\alpha}(\mathbf{k}^*-\mathbf{j}^*,t)u^{<*}_{\beta}(\mathbf{j}^*-\mathbf{j}'^*)T^{<*}(\mathbf{j},t) \ ,$$

since j^* is in the subgrid while k^*,j'^* are in the resolvable scale. Thus the $\delta(\mathbf{k}^*-\mathbf{j}^*)$ and $\delta(\mathbf{j}^*-\mathbf{j}'^*)$ can never be satisfied simultaneously.

The second triple nonlinear term, labeled $\mathcal{P}_2$, has the following structure after the Galilean transformation.

$$\mathcal{P}_2\sim\int d^3j^* d^3j'^*[u^{<*}_{\beta}(\mathbf{j}'^*,t)+U_{0\beta}\delta(\mathbf{j}'^*)][u^{<*}_{\gamma}(\mathbf{k}^*-\mathbf{j}^*-\mathbf{j}'^*,t)+U_{0\gamma}\delta(\mathbf{k}^*-\mathbf{j}^*-\mathbf{j}'^*)]T^{<}(\mathbf{j},t)$$
$$\rightarrow\int d^3j^* d^3[u^{<*}_{\beta}(\mathbf{j}'^*)+U_{0\beta}\delta(\mathbf{j}'^*)][u^{<*}_{\gamma}(\mathbf{k}^*-\mathbf{j}^*-\mathbf{j}'^*,t)T^{<}(\mathbf{j},t)$$
$$\rightarrow\int d^3j^* d^3 u^{<*}_{\beta}(\mathbf{j}'^*,t)u^{<*}_{\gamma}(\mathbf{k}^*-\mathbf{j}^*-\mathbf{j}'^*,t)T^{<}(\mathbf{j},t) \ ,$$

where the last two steps follow from the wave-number constraints. k,j,j' are in the resolvable scales while $|\mathbf{k}-\mathbf{j}|$ is in the subgrid scale. Specifically, the first step follows since $\delta(\mathbf{k}^*-\mathbf{j}^*-\mathbf{j}'^*)$ can never be satisfied. The second step follows since $\delta(\mathbf{j}'^*)$ would force $u^{<*}_{\gamma}(\mathbf{k}^*-\mathbf{j}^*-\mathbf{j}'^*)\rightarrow u^{<*}_{\gamma}(\mathbf{k}^*-\mathbf{j}^*)$. This is not permissible since $|\mathbf{k}-\mathbf{j}|$ is in the subgrid while u^*_{γ}, by definition, is in the resolvable scale. Thus the renormalized passive scalar equation is also Galilean invariant.

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